MATH2050B Mathematical Analysis I

End of term Make-up Test suggested Solution[*]

Question 1. Let $f:\mathbb{R}\to\mathbb{R}$ and (x_n) a seq of real number s (unless explicitly otherwise), Q1 State each of the following definitions/notations.

(a) $\lim\limits_{n} x_n = -\infty$.

(b) $\lim\limits_{x\to+\infty} f(x) = \ell\ (\in \mathbb{R})$.

(c) $\lim\limits_{x\to a+} f(x) = \ell\ (a, \ell \in \mathbb{R})$.

(d) f is continuous at x_0 $(x_0 \in \mathbb{R})$.

(e) f is uniformly continuous on $\mathbb{R}$.

(f) $\text{limsup}_n\, x_n$.

State the negation for (d) and the negation for (e).

Solution:

(a) For any $M\in\mathbb{R}$, there exists $N\in\mathbb{N}$ such that $x_n < M$ for all $n \geq N$.

(b) For any $\varepsilon > 0$, there exists $M\in\mathbb{R}$ such that $|f(x)-\ell| < \varepsilon$ for all $x \geq M$.

(c) For any $\varepsilon > 0$, there exists $\delta > 0$ such that for any $x\in(a, a+\delta)$,

$$|f(x)-\ell| < \varepsilon.$$

(d) For any $\varepsilon > 0$, there exists $\delta > 0$ such that for any $x\in(x_0-\delta, x_0+\delta)$,

$$|f(x)-f(x_0)| < \varepsilon.$$

Negation: There exists an $\varepsilon_0 > 0$ such that for every $\delta > 0$ there exists $x_\delta \in (x_0-\delta, x_0+\delta)$ satisfying $|f(x_\delta)-f(x_0)| \geq \varepsilon_0$.

(e) For any $\varepsilon > 0$, there is a $\delta(\varepsilon) > 0$ such that if $x, u\in\mathbb{R}$ are any numbers satisfying $|x-u| < \delta(\varepsilon)$, then $|f(x)-f(u)| < \varepsilon$.

Negation: There exists an $\varepsilon_0 > 0$ such that for every $\delta > 0$ there are points x_δ, u_δ in $\mathbb{R}$ such that $|x_\delta - u_\delta| < \delta$ and $|f(x_\delta) - f(u_\delta)| \geq \varepsilon_0$.

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(f) The limit superior of (x_n) is the infimum of the set V of all $v \in \mathbb{R}$ which satisfies that there exists $N(v) \in \mathbb{N}$ such that $x_n \leq v$ for all $n \geq N(v)$.

Question 2. State each of the following results/theorems:

(a) The well-order properties result for $\mathbb{Z}$ in $\mathbb{R}$.

(b) The interval charaterization theorem.

(c) The nested interval theorem.

(d) The Monotone Convergence Theorem for seq.

(e) The Monotone Convergene Theorm for functions.

(f) The max-min value theorem.

(g) The root theorem (or the Intermediate-Value Th.)

(h) The uniform continuity theorem.

(i) An order-preserving result for seq.

(j) An order-preserving result for functions.

Solution:

(a) Any non-empty bounded below subset of $\mathbb{Z}$ has a least element.

(b) If S is a subset of $\mathbb{R}$ that contains at least two points and has the property

$$\text{if } x, y \in S \text{ and } x < y, \text{ then } [x, y] \subseteq S,$$

then S is an interval.

(c) If $I_n = [a_n, b_n], n \in \mathbb{N}$, is a nested sequence of closed bounded intervals, then there exists a number $\xi \in \mathbb{R}$ such that $\xi \in I_n$ for all $n \in \mathbb{N}$.

(d) A monotone sequence of real numbers is convergent if and only if it is bounded. Further:

(i) If $X = (x_n)$ is a bounded increasing sequence, then

$$\lim (x_n) = \sup \{x_n : n \in \mathbb{N}\}.$$

(ii) If $Y = (y_n)$ is a bounded decreasing sequence, then

$$\lim (y_n) = \inf \{y_n : n \in \mathbb{N}\}.$$

(e) Let $f : I \to \mathbb{R}$ be increasing and $(a, b) \subseteq I$. Then

(i) $\lim_{x \to a+} f = \inf\{f(x) : x \in I, x > a\}$,

(ii) $\lim_{x\to b-} f = \sup\{f(x) : x \in I, x < b\}$.

The corresponding result for decreasing functions also holds.

(f) Let $I := [a, b]$ be a closed bounded interval and let $f : I \to \mathbb{R}$ be continuous on I. Then f has an absolute maximum and an absolute minimum on I.

(g) Let $I = [a, b]$ and let $f : I \to \mathbb{R}$ be continuous on I. If $f(a) < 0 < f(b)$, or if $f(a) > 0 > f(b)$, then there exists a number $c \in (a, b)$ such that $f(c) = 0$.

(h) Let I be a closed bounded interval and let $f : I \to \mathbb{R}$ be continuous on I. Then f is uniformly continuous on I.

(i) Let (x_n) be a convergent sequence in $\mathbb{R}$. If $a \le x_n \le b$ for all $n \in \mathbb{N}$, then $a \le \lim\limits_{n\to\infty} x_n \le b$. Also, if $\alpha < \lim\limits_{n} y_n < \beta$ then there exists $N \in \mathbb{N}$ such that $\alpha < y_n < \beta$ for all $n \ge N$.

(j) Let $A \subseteq \mathbb{R}$, let $f : A \to \mathbb{R}$, and let $c \in \mathbb{R}$ be a cluster point of A. If

$$a \le f(x) \le b \text{ for all } x \in A, x \ne c,$$

and if $\lim\limits_{x\to c} f(x)$ exists, then $a \le \lim\limits_{x\to c} f(x) \le b$.

Question 3. State and prove the Bolzano-Weierstrass Th. Yon may make use any results in $Q2$. In particular you may wish to apply (b) and the bisection technique (Hint: any seq (x_n) in $[a, b] = I \cup J \Rightarrow$ either I or J contains x_n for infinitely many n). If yon make use of (e) then you must attach a proof of the existence for a monotone subsequence.

Solution:

Bolzano-Weierstrass Theorem: A bounded sequence of real numbers has a convergent subsequence.

First Proof. Since the set of values $\{x_n : n \in \mathbb{N}\}$ is bounded, this set is contained in an interval $I_1 := [a, b]$. We take $n_1 := 1$.

We now bisect I_1 into two equal subintervals I_1' and I_1'', and divide the set of indices $\{n \in \mathbb{N} : n > 1\}$ into two parts:

$$A_1 := \{n \in \mathbb{N} : n > n_1, x_n \in I_1'\}, \quad B_1 = \{n \in \mathbb{N} : n > n_1, x_n \in I_1''\}.$$

If A_1 is infinite, we take $I_2 := I_1'$ and let n_2 be the smallest natural number in A_1. If A_1 is a finite set, then B_1 must be infinite, and we take $I_2 := I_1''$ and let n_2 be the smallest natural number in B_1.

We now bisect I_2 into two equal subintervals I_2' and I_2'', and divide the set $\{n \in \mathbb{N} : n > n_2\}$ into two parts:

$$A_2 = \{n \in \mathbb{N} : n > n_2, x_n \in I_2'\}, \quad B_2 := \{n \in \mathbb{N} : n > n_2, x_n \in I_2''\}.$$

If A_2 is infinite, we take $I_3 := I_2'$ and let n_3 be the smallest natural number in A_2. If A_2 is a finite set, then B_2 must be infinite, and we take $I_3 := I_2''$ and let n_3 be the smallest natural number in B_2.

We continue in this way to obtain a sequence of nested intervals $I_1 \supseteq I_2 \supseteq \cdots \supseteq I_k \supseteq \cdots$ and a subsequence (x_{n_k}) of X such that $x_{n_k} \in I_k$ for $k \in \mathbb{N}$. Since the length of I_k is equal to $(b-a)/2^{k-1}$, it follows from Nested Intervals Theorem that there is a (unique) common point $\xi \in I_k$ for all $k \in \mathbb{N}$. Moreover, since x_{n_k} and ξ both belong to I_k, we have

$$|x_{n_k} - \xi| \leq (b-a)/2^{k-1}$$

whence it follows that the subsequence (x_{n_k}) of X converges to ξ.

Second Proof. Firstly, we show that any bounded sequence (x_n) has a subsequence that is monotone. We will call x_m a peak if $n \geq m \Rightarrow x_n \leq x_m$ (i.e, if no term to the right of x_m is greater than x_m).

Case 1: X has infinitely many peaks. Order the peaks by increasing subscripts. Then

$$x_{m_1} \geq x_{m_2} \geq \cdots \geq x_{m_k} \geq \cdots,$$

so $\{x_{m_k}\}$ is a decreasing subsequence.

Case 2: X has finitely many (maybe 0) peaks. Let $x_{m_1}, x_{m_2}, \ldots, x_{m_r}$ denote these peaks. Let $s_1 = m_r + 1$ (the first index past the last peak) or $s_1 = 1$ if there are no peaks. Since x_{s_1} is not a peak, there exists $s_2 > s_1$ such that $x_{s_1} < x_{s_2}$. Since x_{s_2} is not a peak, there exists $s_3 > s_2$ such that $x_{s_2} < x_{s_3}$. Continuing, we get an increasing subsequence.

It follows that if $X = (x_n)$ is a bounded sequence, then it has a subsequence X' that is monotone. Since this subsequence is also bounded, it follows from the Monotone Convergence that the subsequence is convergent.

Question 4. Suppose $1 < r < \liminf\limits_{n} x_n^{1/n} \in \mathbb{R}$, where $n \in \mathbb{N}$. Show that $\exists N \in \mathbb{N}$ s.t.

$$r < x_n^{1/n} \quad \forall n \geqslant N,$$

and that $\sum\limits_{n=1}^{\infty} x_n = +\infty$.

Solution: Let $a = \liminf\limits_{n} x_n^{1/n}$ and $\varepsilon = \frac{a-r}{2}$. It follows from the definition of Limit Inferior that there exists $N \in \mathbb{N}$ such that

$$a - \varepsilon < x_n^{1/n}, \quad \forall n \geqslant N.$$

This implies that $x_n^{1/n} > r$, due to the fact that $a - \frac{a-r}{2} > r$. It also yields that $x_n > r^n$ for all $n \geqslant N$.

Notice that the geometric series $\sum_{n=1}^{\infty} r^n = \infty$ is divergent, since $r > 1$. By Comparison Test, we obtain $\sum_{n=1}^{\infty} x_n = \infty$.

Question 5. In $\varepsilon - \delta$ (or $\varepsilon - N$) terminology show that

(a) If $\lim_n x_n = x \in \mathbb{R}$ and $\lim_n y_n = y$ then

$$\lim_n x_n y_n = xy.$$

(b) $\lim_{x\to 2} \frac{x^2+8}{x^2-1} = 4$.

(c) Let $f, g, F, G : (0, +\infty) \to \mathbb{R}$ be such that

$$\lim_{x\to\infty} F(x) = L_1, \quad \lim_{x\to\infty} G(x) = L_2 \quad (L_1, L_2 \in \mathbb{R} \setminus \{0\})$$

$$\lim_{x\to\infty} f(x) = +\infty = \lim_{x\to\infty} g(x), \quad \lim_{x\to\infty} (f(x)/x) = \ell_1, \quad \lim_{x\to\infty} (g(x)/x) = \ell_2 \quad (\ell_1, \ell_2 \in \mathbb{R} \setminus \{0\}).$$

Then

$$\lim_{x\to\infty} (F(x)/G(x)) = \frac{L_1}{L_2}, \quad \text{and} \quad \lim_{x\to\infty} (f(x)/g(x)) = \frac{\ell_1}{\ell_2}.$$

Solution:

(a) Since $\lim_n y_n = y$, there exists $N_0 \in \mathbb{N}$ such that for any $n \geq N_0$, we have $|y_n - y| < 1$. It directly follows that $|y_n| < |y| + 1$ for any $n \geq N_0$.

Fix $\varepsilon > 0$. Take $\varepsilon' > 0$ such that $\varepsilon' = \min\{\frac{\varepsilon}{(|x|+|y|+2)}, 1\}$. Since $\lim_n x_n = x$, there exists $N_1(\varepsilon) \in \mathbb{N}$ such that for any $n \geq N_1(\varepsilon)$, we have $|x_n - x| < \varepsilon'$. Similarly, since $\lim_n y_n = y$, there exists $N_2(\varepsilon) \in \mathbb{N}$ such that for any $n \geq N_2(\varepsilon)$, we obtain $|y_n - y| < \varepsilon'$.

Hence, the triangle inequality implies that

$$\begin{aligned}
|x_n y_n - xy| &\leq |x_n y_n - x y_n| \\
&\leq |x_n - x|\,|y_n| + |x|\,|y_n - y| \\
&< \varepsilon' \cdot (|y| + 1) + (|x| + 1)\varepsilon' \\
&= (|x| + |y| + 2)\varepsilon' \\
&\leq \varepsilon,
\end{aligned}$$

for all $n \geq \max\{N_0, N_1(\varepsilon), N_2(\varepsilon)\}$. This implies that $(x_n y_n)$ is convergent and $\lim x_n y_n = xy$.

(b) Let $\varepsilon > 0$, take $\delta(\varepsilon) = \min\{\frac{1}{2}, \frac{\varepsilon}{16}\}$.

Suppose $|x - 2| < \delta(\varepsilon)$, then

$$-\frac{1}{2} < x - 2 < \frac{1}{2} \quad \text{i.e.} \quad \frac{3}{2} < x < \frac{5}{2},$$

which implies that $x^2 - 1 > 1$ and $|x+2| < 5$.

It follows that

$$\begin{aligned}\left|\frac{x^2+8}{x^2-1} - 4\right| &= \left|\frac{x^2+8-4(x^2-1)}{x^2-1}\right| \\ &= \left|\frac{3(x^2-4)}{x^2-1}\right| \\ &= 3\frac{|(x+2)(x-2)|}{|x^2-1|} \\ &< 3\frac{5}{1} \cdot |(x-2)| \\ &< 15 \cdot \delta(\varepsilon) \\ &< \varepsilon.\end{aligned}$$

Therefore $\lim\limits_{x\to 2} \frac{x^2+8}{x^2-1} = 4$.

(c) Fix $\varepsilon > 0$, and let $\epsilon' = \{\frac{|L_2|^2}{2(|L_1|+|L_2|)}\varepsilon, 1\}$. Since $\lim\limits_{x\to\infty} F(x) = L_1$, there exists $M_1 \in \mathbb{R}$ such that $|F(x) - L_1| < \epsilon'$ for all $x \geq M_1$. Similarly, since $\lim\limits_{x\to\infty} G(x) = L_2$, there exists $M_2 \in \mathbb{R}$ such that $|F(x) - L_2| < \epsilon'$ for all $x \geq M_2$.

Let $M(\varepsilon) = \max\{M_1, M_2\}$. Then for any $x \geq M(\varepsilon)$,

$$\begin{aligned}\left|\frac{F(x)}{G(x)} - \frac{L_1}{L_2}\right| &= \left|\frac{L_2F(x) - L_1G(x)}{L_2G(x)}\right| \\ &= \left|\frac{L_2F(x) - L_2L_1 + L_2L_1 - L_1G(x)}{L_2G(x)}\right| \\ &\leq \frac{|L_2||F(x) - L_1| + |L_1| \cdot |G(x) - L_2|}{|L_2G(x)|} \\ &\leq \frac{|L_2|\epsilon'}{|L_2| \cdot (|L_2|/2)} + \frac{|L_1|\epsilon'}{|L_2| \cdot (|L_2|/2)} \\ &= \frac{2\,(|L_1| + |L_2|)}{|L_2|^2}\epsilon' \\ &\leq \varepsilon,\end{aligned}$$

that is, $\lim\limits_{x\to\infty} \dfrac{F(x)}{G(x)} = \dfrac{L_1}{L_2}$.

Next we show that $\lim_{x\to\infty}(f(x)/g(x)) = \dfrac{\ell_1}{\ell_2}$. For $x > 0$, we define $\widetilde{F}(x) := \frac{f(x)}{x}$ and $\widetilde{G}(x) := \frac{g(x)}{x}$. By above assumption we have $\lim\limits_{x\to\infty} \widetilde{F}(x) = \ell_1$ and $\lim\limits_{x\to\infty} \widetilde{G}(x) = \ell_2$.

Notice that

$$\frac{f(x)}{g(x)} = \frac{f(x)}{x} \cdot \frac{x}{g(x)} = \frac{\widetilde{F}(x)}{\widetilde{G}(x)}, \qquad \text{for all } x > 0.$$

Since $\lim_{x\to\infty} \widetilde{F}(x) = \ell_1$ and $\lim_{x\to\infty} \widetilde{G}(x) = \ell_2$ with $\ell_1, \ell_2 \neq 0$, it follows by above result that

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = \lim_{x\to\infty} \frac{\widetilde{F}(x)}{\widetilde{G}(x)} = \frac{\ell_1}{\ell_2}.$$

Remark: The method to prove $\lim_{x\to\infty} \dfrac{\widetilde{F}(x)}{\widetilde{G}(x)} = \dfrac{\ell_1}{\ell_2}$ is similar to that used in our previous argument.